

Green's Function of a generalized boundary value transmission problem

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Abstract

The aim of this study is to investigate a class of generalized boundary value transmission problems (BVTP's) for Sturm-Liouville equation on two separate intervals. We introduce modified inner product in direct sum space $L_2[a, c] \oplus L_2(c, b] \oplus \mathbb{C}^2$ and define symmetric linear operator in it such a way that the considered problem can be interpreted as an eigenvalue problem of this operator. Then by suggesting an own approaches we construct Green's function for problem under consideration and find the resolvent function for corresponding inhomogeneous problem.

Keywords: Sturm-Liouville problems, Green's function, transmission conditions, resolvent operator.

AMS subject classifications : 34B24, 34B27

1. Introduction

Boundary value problems can be investigate through the methods of Green's function and eigenfunction expansion. The main tool for solvability analysis of such problems is the concept of Green's function. The concept of Green's functions is very close to physical intuition (see [1]). If one knows the Green's function of a problem one can write down its solution in closed form as linear combinations of integrals involving the Green's function and the functions appearing in the inhomogeneities. Green's functions can often be found in an explicit way, and in these cases it is very efficient to solve

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the problem in this way. Determination of Greens functions is also possible using Sturm-Liouville theory. This leads to series representation of Green's functions (see [3]). In this paper we shall investigate a class of BVP's which consist of the Sturm-Liouville equation

$$\mathcal{L}(u) := -\rho(x)u''(x) + q(x)u(x) = \lambda u(x), x \in \Omega \quad (1)$$

together with eigenparameter-dependent boundary conditions at end points $x = a, b$

$$\mathcal{L}_1(u) := \delta_1 u(a) - \delta_2 u'(a) - \lambda(\delta_3 u(a) - \delta_4 u'(a)) = 0, \quad (2)$$

$$\mathcal{L}_2(u) := \gamma_1 u(b) - \gamma_2 u'(b) + \lambda(\gamma_3 u(b) - \gamma_4 u'(b)) = 0, \quad (3)$$

and the transmission conditions at interior point $\xi_i \in (a, b), i = 1, 2, \dots, n$

$$\mathcal{L}_i(u) = \delta_{i1}^- u'(\xi_i -) + \delta_{i0}^- u(\xi_i -) + \delta_{i1}^+ u'(\xi_i +) + \delta_{i0}^+ u(\xi_i +) = 0, \quad (4)$$

$$\mathcal{L}_i(u) = \gamma_{i1}^- u'(\xi_i -) + \gamma_{i0}^- u(\xi_i -) + \gamma_{i1}^+ u'(\xi_i +) + \gamma_{i0}^+ u(\xi_i +) = 0, \quad (5)$$

where $\Omega = \bigcup_{i=1}^{n+1} (\xi_{i-1}, \xi_i)$, $a := \xi_0$, $b := \xi_{n+1}$, $\rho(x) = \rho_i^2 > 0$ for $x \in \Omega_i := (\xi_{i-1}, \xi_i)$, $i = 1, 2, \dots, n+1$, the potential $q(x)$ is real-valued function which continuous in each of the intervals (ξ_{i-1}, ξ_i) , and has a finite limits $q(\xi_i \mp 0)$, λ is a complex spectral parameter, δ_k, γ_k ($k = 1, 2, 3, 4$), $\delta_{ij}^\pm, \gamma_{ij}^\pm$ ($i = 1, 2, \dots, n$ and $j = 0, 1$) are real numbers. We want emphasize that the boundary value problem studied here differs from the standard boundary value problems in that it contains transmission conditions and the eigenvalue-parameter appears not only in the differential equation, but also in the boundary conditions. Moreover the coefficient functions may have discontinuity at one interior point. Naturally, eigenfunctions of this problem may have discontinuity at the one inner point of the considered interval. The problems with transmission conditions has become an important area of research in recent years because of the needs of modern technology, engineering and physics. Many of the mathematical problems encountered in the study of boundary-value-transmission problem cannot be treated with the usual techniques within the standard framework of boundary value problem (see [2]). Note that some special cases of this problem arise after an application of the method of separation of variables to a varied assortment of physical problems. For example, some boundary value problems with transmission conditions arise in heat and mass transfer problems [4], in vibrating

string problems when the string loaded additionally with point masses [10], in diffraction problems [13]. Also some problems with transmission conditions which arise in mechanics (thermal conduction problems for a thin laminated plate) were studied in [12].

2. The „basic” solutions and characteristic function

We shall define two solutions of the equation (1) on the whole $\Omega = \bigcup_{i=1}^{n+1} (\xi_{i-1}, \xi_i)$ by $\phi(x, \lambda) = \phi_i(x, \lambda)$ for $x \in \Omega_i$ and $\chi(x, \lambda) = \chi_i(x, \lambda)$ for $x \in \Omega_i (i = 1, 2, \dots, n+1)$

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), x \in [a, \xi_1) \\ \phi_2(x, \lambda), x \in (\xi_1, \xi_2) \\ \phi_3(x, \lambda), x \in (\xi_2, \xi_3) \\ \dots \\ \phi_{n+1}(x, \lambda), x \in (\xi_n, b] \end{cases} \quad \text{and} \quad \chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), x \in [a, \xi_1) \\ \chi_2(x, \lambda), x \in (\xi_1, \xi_2) \\ \chi_3(x, \lambda), x \in (\xi_2, \xi_3) \\ \dots \\ \chi_{n+1}(x, \lambda), x \in (\xi_n, b] \end{cases}$$

where $\phi_i(x, \lambda)$ and $\chi_i(x, \lambda)$ are defined recurrently by following procedure. Let $\phi_1(x, \lambda)$ and $\chi_{n+1}(x, \lambda)$ be solutions of the equation (1) on $[a, \xi_1)$ and $(\xi_n, b]$ satisfying initial conditions

$$u(a, \lambda) = \delta_2 - \lambda \delta_4, \quad u'(a, \lambda) = \delta_1 - \lambda \delta_3 \quad (6)$$

$$u(b, \lambda) = \gamma_2 + \lambda \gamma_4, \quad u'(b, \lambda) = \gamma_1 - \lambda \gamma_3 \quad (7)$$

respectively. In terms of these solution we shall define the other solutions $\phi_{i+1}(x, \lambda)$ and $\chi_i(x, \lambda)$ by initial conditions

$$\phi_{i+1}(\xi_i+, \lambda) = \frac{1}{\theta_{i12}}(\theta_{i23}\phi_i(\xi_i-, \lambda) + \theta_{i24}\frac{\partial \phi_i(\xi_i-, \lambda)}{\partial x}) \quad (8)$$

$$\frac{\partial \phi_{i+1}(\xi_i+, \lambda)}{\partial x} = \frac{-1}{\theta_{i12}}(\theta_{i13}\phi_i(\xi_i-, \lambda) + \theta_{i14}\frac{\partial \phi_i(\xi_i-, \lambda)}{\partial x}) \quad (9)$$

and

$$\chi_i(\xi_i-, \lambda) = \frac{-1}{\theta_{i34}}(\theta_{i14}\chi_{i+1}(\xi_i+, \lambda) + \theta_{i24}\frac{\partial \chi_{i+1}(\xi_i+, \lambda)}{\partial x}) \quad (10)$$

$$\frac{\partial \chi_i(\xi_i-, \lambda)}{\partial x} = \frac{1}{\theta_{i34}}(\theta_{i13}\chi_{i+1}(\xi_i+, \lambda) + \theta_{i23}\frac{\partial \chi_{i+1}(\xi_i+, \lambda)}{\partial x}) \quad (11)$$

respectively, where θ_{ijk} ($1 \leq j < k \leq 4$) denotes the determinant of the j -th and k -th columns of the matrix

$$\begin{bmatrix} \delta_{i1}^+ & \delta_{i0}^+ & \delta_{i1}^- & \delta_{i0}^- \\ \gamma_{i1}^+ & \gamma_{i0}^+ & \gamma_{i1}^- & \gamma_{i0}^- \end{bmatrix}$$

for $i = 1, 2, \dots, n$. Everywhere in below we shall assume that $\theta_{ijk} > 0$ for all i, j, k . The existence and uniqueness of these solutions are follows from well-known theorem of ordinary differential equation theory. Moreover by applying the method of [7] we can prove that each of these solutions are entire functions of parameter $\lambda \in \mathbb{C}$ for each fixed x . Taking into account (8)-(11) and the fact that the Wronskians $\omega_i(\lambda) := W[\phi_i(x, \lambda), \chi_i(x, \lambda)]$ ($i=1, 2, \dots, n+1$) are independent of variable x we have

$$\begin{aligned} \omega_{i+1}(\lambda) &= \phi_{i+1}(\xi_i+, \lambda) \frac{\partial \chi_{i+1}(\xi_i+, \lambda)}{\partial x} - \frac{\partial \phi_{i+1}(\xi_i+, \lambda)}{\partial x} \chi_{i+1}(\xi_i+, \lambda) \\ &= \frac{\theta_{i34}}{\theta_{i12}} (\phi_i(\xi_i-, \lambda) \frac{\partial \chi_i(\xi_i, \lambda)}{\partial x} - \frac{\partial \phi_i(\xi_i-, \lambda)}{\partial x} \chi_i(\xi_i-, \lambda)) \\ &= \frac{\theta_{i34}}{\theta_{i12}} \omega_i(\lambda) = \prod_{j=1}^i \frac{\theta_{j34}}{\theta_{j12}} \omega_1(\lambda) \quad (i = 1, 2, \dots, n). \end{aligned}$$

It is convenient to define the characteristic function $\omega(\lambda)$ for our problem (1) – (3) as

$$\omega(\lambda) := \omega_1(\lambda) = \prod_{j=1}^i \frac{\theta_{j12}}{\theta_{j34}} \omega_{i+1}(\lambda) \quad (i = 1, 2, \dots, n).$$

Obviously, $\omega(\lambda)$ is an entire function. By applying the technique of [8] we can prove that there are infinitely many eigenvalues λ_k , $k = 1, 2, \dots$ of the problem (1) – (5) which are coincide with the zeros of characteristic function $\omega(\lambda)$.

3. Operator treatment in modified Hilbert space

To analyze the spectrum of the BVTP (1) – (5) we shall construct an adequate Hilbert space and define a symmetric linear operator in it such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. For this we assume that

$$\kappa_1 := \begin{bmatrix} \delta_3 & \delta_4 \\ \delta_1 & \delta_2 \end{bmatrix} > 0, \quad \kappa_2 := \begin{bmatrix} \gamma_3 & \gamma_4 \\ \gamma_1 & \gamma_2 \end{bmatrix} > 0$$

and introduce modified inner product on direct sum space $\mathcal{H}_1 = L_2(a, \xi_1) \oplus L_2(\xi_1, \xi_2) \oplus \dots \oplus L_2(\xi_{n-1}, \xi_n) \oplus L_2(\xi_n, b)$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathbb{C}^2$ by

$$\langle f, g \rangle_{\mathcal{H}_1} := \sum_{k=0}^n \frac{1}{\rho_{k+1}^2} \prod_{i=0}^k \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+}^{\xi_{k+1}-} f(x) \overline{g(x)} dx$$

where $\theta_{034} = \theta_{(n+1)12} = 1$ and

$$\langle F, G \rangle_{\mathcal{H}} := [f, g]_{\mathcal{H}_1} + \prod_{i=0}^n \theta_{i12} \frac{f_1 \overline{g_1}}{\kappa_1} + \prod_{i=0}^n \theta_{i34} \frac{f_2 \overline{g_2}}{\kappa_2}$$

for $F = (f(x), f_1, f_2)$, $G = (g(x), g_1, g_2) \in \mathcal{H}$ respectively. Obviously, these inner products are equivalent to the standard inner products, so, $(\mathcal{H}, [\cdot, \cdot]_{\mathcal{H}})$ and $(\mathcal{H}_1, [\cdot, \cdot]_{\mathcal{H}_1})$ are also Hilbert spaces. Let us now define the boundary functionals

$$\begin{aligned} B_a[f] &:= \delta_1 f(a) - \delta_2 f'(a), & B'_a[f] &:= \delta_3 f(a) - \delta_4 f'(a) \\ B_b[f] &:= \gamma_1 f(b) - \gamma_2 f'(b), & B'_b[f] &:= \gamma_3 f(b) - \gamma_4 f'(b) \end{aligned}$$

and construct the operator $\mathfrak{R} : \mathcal{H} \rightarrow \mathcal{H}$ with the domain

$$\begin{aligned} \text{dom}(\mathfrak{R}) := & \left\{ F = (f(x), f_1, f_2) : f(x), f'(x) \in \bigcap_{i=1}^{i=n+1} AC_{loc}(\xi_{i-1}, \xi_i), \right. \\ & \text{and has a finite limits } f(\xi_i \mp 0) \text{ and } f'(\xi_i \mp 0), \mathcal{L}F \in L_2[a, b], \\ & \left. \mathcal{L}_i(f) = \mathfrak{L}(f) = 0, f_1 = B'_a[f], f_2 = -B'_b[f] \right\} \end{aligned}$$

and action low

$$\mathcal{L}(f(x), B'_a[f], -B'_b[f]) = (\ell f, B_a[f], B_b[f]).$$

Then the problem (1) – (5) can be written in the operator equation form as

$$\mathfrak{R}F = \lambda F, \quad F = (f(x), B'_a[f], -B'_b[f]) \in \text{dom}(\mathfrak{R})$$

in the Hilbert space \mathcal{H} .

Theorem 1. *The linear operator \mathfrak{R} is symmetric.*

PROOF. By applying the method of [8] it is not difficult to show that $dom(\mathfrak{R})$ is dense in the Hilbert space \mathcal{H} . Now let $F = (f(x), B'_a[f], -B'_b[f]), G = (g(x), B'_a[g], -B'_b[g]) \in dom(\mathfrak{R})$. By partial integration we have

$$\begin{aligned}
& \langle \mathfrak{R}F, G \rangle_{\mathcal{H}} - \langle F, \mathfrak{R}G \rangle_{\mathcal{H}} = \theta_{112}\theta_{212}\dots\theta_{n12}(W(f, \bar{g}; \xi_1-) - W(f, \bar{g}; a)) \\
& + \theta_{134}\theta_{212}\dots\theta_{n12}(W(f, \bar{g}; \xi_2-) - W(f, \bar{g}; \xi_1+)) \\
& + \dots + \theta_{134}\theta_{234}\dots\theta_{n34}(W(f, \bar{g}; b) - W(f, \bar{g}; \xi_n+)) \\
& + \frac{1}{\kappa_1} \prod_{i=0}^n \theta_{i12} (B_a[f] \overline{B'_a[g]} - B'_a[f] \overline{B_a[g]}) \\
& + \frac{1}{\kappa_2} \prod_{i=0}^n \theta_{i34} (B'_b[f] \overline{B_b[g]} - B_b[f] \overline{B'_b[g]}) \tag{12}
\end{aligned}$$

where, as usual, $W(f, \bar{g}; x)$ denotes the Wronskians of the functions f and \bar{g} . From the definitions of boundary functionals we get that

$$B_a[f] \overline{B'_a[g]} - B'_a[f] \overline{B_a[g]} = \kappa_1 W(f, \bar{g}; a), \tag{13}$$

$$B'_b[f] \overline{B_b[g]} - B_b[f] \overline{B'_b[g]} = -\kappa_2 W(f, \bar{g}; b) \tag{14}$$

Further, taking in view the definition of \mathcal{L} and initial conditions (6) – (11) we derive that

$$\theta_{i12} W(f, \bar{g}; \xi_i-) = \theta_{i34} W(f, \bar{g}; \xi_i+) \quad i = 1, 2, \dots, n \tag{15}$$

Finally, substituting (13), (14) and (15) in (12) we have

$$\langle \mathfrak{R}F, G \rangle_{\mathcal{H}} = \langle F, \mathfrak{R}G \rangle_{\mathcal{H}} \quad \text{for every } F, G \in dom(\mathfrak{R}),$$

so the operator \mathfrak{R} is symmetric in \mathcal{H} . The proof is complete.

Theorem 2. *The operator \mathfrak{R} is self-adjoint in \mathcal{H} .*

Corollary 1. *If $f(x)$ and $g(x)$ are eigenfunctions corresponding to distinct eigenvalues, then they are „orthogonal” in the sense of*

$$\langle f, g \rangle_{\mathcal{H}_1} + \prod_{i=0}^n \theta_{i12} \frac{B'_a[f] B'_a[g]}{\kappa_1} + \prod_{i=0}^n \theta_{i34} \frac{B'_b[f] B'_b[g]}{\kappa_2} = 0 \tag{16}$$

where $F = (f(x), B'_a[f], -B'_b[f]), G = (g(x), B'_a[g], -B'_b[g]) \in dom(\mathfrak{R})$.

Theorem 3. *The operator \mathfrak{R} has only point spectrum, i.e. $\sigma(\mathfrak{R}) = \sigma_p(\mathfrak{R})$.*

Lemma 1. *The operator \mathfrak{R} has compact resolvent, i.e. for $\forall \delta \in \mathbb{R}/\sigma_p(\mathfrak{R})$, $(\mathfrak{R} - \delta I)$ is compact in \mathcal{H} .*

By applying the above results, we obtain the following theorem.

Theorem 4. *The eigenfunctions of the problem (1) – (5), augmented to become eigenfunctions of \mathfrak{R} , are complete in \mathcal{H} , i.e. if we let $\{\Psi_s = (\Psi_s(x), B'_a[\Psi_s], -B'_b[\Psi_s]); n \in N\}$ be a maximum set of orthonormal eigenfunctions of \mathfrak{R} , where $\{\Psi_s(x); s \in N\}$ are eigenfunctions of the problem (1) – (5), then for all $(f(x), f_1, f_2) \in \mathcal{H}$,*

$$\begin{aligned} f &= \sum_{s=1}^{\infty} \left\{ \sum_{k=0}^n \frac{1}{\rho_{k+1}^2} \prod_{i=0}^k \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+}^{\xi_{k+1}-} f(x) \Psi_s(x) dx + \prod_{i=0}^n \theta_{i12} \frac{f_1 B'_a[\Psi_s]}{\kappa_1} \right. \\ &\quad \left. + \prod_{i=0}^n \theta_{i34} \frac{f_2 (-B'_b[\Psi_s])}{\kappa_2} \right\} \Psi_s. \end{aligned}$$

4. Solvability of the corresponding inhomogeneous problem

Now let $\lambda \in \mathbb{C}$ not be an eigenvalue of \mathfrak{R} and consider the operator equation

$$(\lambda I - \mathfrak{R})u = f(x), \quad (17)$$

for arbitrary $u = (u(x), u_1, u_2) \in \mathcal{H}$. This operator equation is equivalent to the following inhomogeneous BVTP

$$(\lambda - \mathfrak{R})u(x) = f(x), \quad x \in \Omega \quad (18)$$

$$\mathcal{L}_i(u) = \mathfrak{L}_i(u) = 0, \quad \mathcal{L}_1(u) = 0, \quad \mathcal{L}_2(u) = 0 \quad (19)$$

We shall search the general solution of the non-homogeneous differential Equation (18) in the form

$$u(x, \lambda) = \begin{cases} f_{11\lambda}(x)\phi_{1\lambda}(x) + f_{12\lambda}(x)\chi_{1\lambda}(x) & \text{for } x \in (a, \xi_1) \\ f_{21\lambda}(x)\phi_{2\lambda}(x) + f_{22\lambda}(x)\chi_{2\lambda}(x) & \text{for } x \in (\xi_1, \xi_2) \\ \dots & \dots \\ f_{i1\lambda}(x)\phi_{i\lambda}(x) + f_{i2\lambda}(x)\chi_{i\lambda}(x) & \text{for } x \in (\xi_n, b) \end{cases} \quad (20)$$

where the functions $f_{ij\lambda}(x) := f_{ij}(x, \lambda)$ ($i = 1, 2, \dots, n+1$, $j = 1, 2$) are the solutions of the system of equations

$$\begin{cases} f'_{11\lambda}(x)\phi_{1\lambda}(x) + f'_{12\lambda}\chi_{1\lambda}(x) = 0 \\ f'_{11\lambda}\phi'_{1\lambda}(x) + f'_{12\lambda}\chi'_{1\lambda}(x) = \frac{f(x)}{\rho_1^2} \end{cases}, \begin{cases} f'_{21\lambda}(x)\phi_{2\lambda}(x) + f'_{22\lambda}(x)\chi_{2\lambda}(x) = 0 \\ f'_{21\lambda}(x)\phi'_{2\lambda}(x) + f'_{22\lambda}(x)\chi'_{2\lambda}(x) = \frac{f(x)}{\rho_2^2} \end{cases}$$

$$\dots\dots \begin{cases} f'_{i1\lambda}(x)\phi_{i\lambda}(x) + f'_{i2\lambda}\chi_{i\lambda}(x) = 0 \\ f'_{i1\lambda}\phi'_{i\lambda}(x) + f'_{i2\lambda}\chi'_{i\lambda}(x) = \frac{f(x)}{\rho_i^2} \end{cases}$$

for $x \in (a, \xi_1)$, $x \in (\xi_1, \xi_2)$, ... $x \in (\xi_i, b)$ ($i = 1, 2, \dots, n+1$), respectively. Since λ is not an eigenvalue $\omega(\lambda) \neq 0$. Then using the transmission conditions we have

$$\begin{aligned} f_{11\lambda}(x) &= \frac{1}{\rho_1^2 \omega_1(\lambda)} \int_x^{\xi_1} u(y) \chi_{1\lambda}(y) dy + f_{11}(\lambda), \quad x \in (a, \xi_1) \\ f_{12\lambda}(x) &= \frac{1}{\rho_1^2 \omega_1(\lambda)} \int_a^x u(y) \phi_{1\lambda}(y) dy + f_{12}(\lambda), \quad x \in (a, \xi_1) \\ f_{21\lambda}(x) &= \frac{1}{\rho_2^2 \omega_2(\lambda)} \int_x^{\xi_2} u(y) \chi_{2\lambda}(y) dy + f_{21}(\lambda), \quad x \in (\xi_1, \xi_2) \\ f_{22\lambda}(x) &= \frac{1}{\rho_2^2 \omega_2(\lambda)} \int_{\xi_1}^x u(y) \phi_{2\lambda}(y) dy + f_{22}(\lambda), \quad x \in (\xi_1, \xi_2) \\ &\dots \\ f_{(n+1)1\lambda}(x) &= \frac{1}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_x^b u(y) \chi_{1\lambda}(y) dy + f_{(n+1)1}(\lambda), \quad x \in (\xi_n, b) \\ f_{(n+1)2\lambda}(x) &= \frac{1}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_{\xi_n}^x u(y) \phi_{i\lambda}(y) dy + f_{(n+1)2}(\lambda), \quad x \in (\xi_n, b) \end{aligned}$$

where $f_{ij}(\lambda)$ ($i, j = 1, 2$) are arbitrary functions of parameter λ . Substituting this into (20) gives

$$u(x, \lambda) = \begin{cases} \frac{\chi_{1\lambda}(x)}{\rho_1^2 \omega_1(\lambda)} \int_a^x \phi_{1\lambda}(y) f(y) dy + \frac{\phi_{1\lambda}(x)}{\rho_1^2 \omega_1(\lambda)} \int_x^{\xi_1} \chi_{1\lambda}(y) f(y) dy \\ + f_{11}(\lambda) \phi_{1\lambda}(x) + f_{12}(\lambda) \chi_{1\lambda}(x) & \text{for } x \in (a, \xi_1) \\ \frac{\chi_{2\lambda}(x)}{\rho_2^2 \omega_2(\lambda)} \int_{\xi_1}^x \phi_{2\lambda}(y) f(y) dy + \frac{\phi_{2\lambda}(x)}{\rho_2^2 \omega_2(\lambda)} \int_x^{\xi_2} \chi_{2\lambda}(y) f(y) dy \\ + f_{21}(\lambda) \phi_{2\lambda}(x) + f_{22}(\lambda) \chi_{2\lambda}(x) & \text{for } x \in (\xi_1, \xi_2) \\ \dots \\ \frac{\chi_{(n+1)\lambda}(x)}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_{\xi_n}^x \phi_{(n+1)\lambda}(y) f(y) dy + \frac{\phi_{(n+1)\lambda}(x)}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_x^b \chi_{(n+1)\lambda}(y) f(y) dy \\ + f_{(n+1)1}(\lambda) \phi_{(n+1)\lambda}(x) + f_{(n+1)2}(\lambda) \chi_{(n+1)\lambda}(x) & \text{for } x \in (\xi_n, b) \end{cases} \quad (21)$$

By differentiating we have

$$u'(x, \lambda) = \begin{cases} \frac{\chi'_{1\lambda}(x)}{\rho_1^2 \omega_1(\lambda)} \int_a^x \phi_{1\lambda}(y) f(y) dy + \frac{\phi'_{1\lambda}(x)}{\rho_1^2 \omega_1(\lambda)} \int_x^{\xi_1} \chi_{1\lambda}(y) f(y) dy \\ + f_{11}(\lambda) \phi'_{1\lambda}(x) + f_{12}(\lambda) \chi'_{1\lambda}(x), & x \in (a, \xi_1) \\ \frac{\chi'_{2\lambda}(x)}{\rho_2^2 \omega_2(\lambda)} \int_{\xi_1}^x \phi_{2\lambda}(y) f(y) dy + \frac{\phi'_{2\lambda}(x)}{\rho_2^2 \omega_2(\lambda)} \int_x^{\xi_2} \chi_{2\lambda}(y) f(y) dy \\ + f_{21}(\lambda) \phi'_{2\lambda}(x) + f_{22}(\lambda) \chi'_{2\lambda}(x), & x \in (\xi_1, \xi_2) \\ \dots \\ \frac{\chi'_{(n+1)\lambda}(x)}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_{\xi_n}^x \phi_{(n+1)\lambda}(y) f(y) dy + \frac{\phi'_{(n+1)\lambda}(x)}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_x^b \chi_{(n+1)\lambda}(y) f(y) dy \\ + f_{(n+1)1}(\lambda) \phi'_{(n+1)\lambda}(x) + f_{(n+1)2}(\lambda) \chi'_{(n+1)\lambda}(x), & x \in (\xi_n, b) \end{cases} \quad (22)$$

By using (21), (22) and the conditions (19) we can derive that $f_{12}(\lambda) = 0$, $f_{(n+1)2}(\lambda) = 0$,

$$\begin{aligned} f_{i1}(\lambda) &= \sum_{s=i+1}^{n+1} \frac{1}{\rho_s^2 \omega_s(\lambda)} \int_{\xi_{s-1}}^{\xi_s} \chi_{s\lambda}(y) f(y) dy, \quad i = 1, 2, \dots, n \\ f_{(i+1)2}(\lambda) &= \sum_{s=1}^i \frac{1}{\rho_s^2 \omega_s(\lambda)} \int_{\xi_{s-1}}^{\xi_s} \phi_{s\lambda}(y) f(y) dy, \quad i = 1, 2, \dots, n \end{aligned}$$

Putting in (21) gives

$$u(x, \lambda) = \begin{cases} \frac{\chi_{1\lambda}(x)}{\rho_1^2 \omega_1(\lambda)} \int_a^x \phi_{1\lambda}(y) f(y) dy + \frac{\phi_{1\lambda}(x)}{\rho_1^2 \omega_1(\lambda)} \int_x^{\xi_1} \chi_{1\lambda}(y) f(y) dy \\ + \phi_{1\lambda}(x) \sum_{s=2}^{n+1} \frac{1}{\rho_s^2 \omega_s(\lambda)} \int_{\xi_{s-1}}^{\xi_s} \chi_{s\lambda}(y) f(y) dy, & x \in (a, \xi_1) \\ \frac{\chi_{2\lambda}(x)}{\rho_2^2 \omega_2(\lambda)} \int_{\xi_1}^x \phi_{2\lambda}(y) f(y) dy + \frac{\phi_{2\lambda}(x)}{\rho_2^2 \omega_2(\lambda)} \int_x^{\xi_2} \chi_{2\lambda}(y) f(y) dy \\ + \phi_{2\lambda}(x) \sum_{s=2}^{n+1} \frac{1}{\rho_s^2 \omega_s(\lambda)} \int_{\xi_{s-1}}^{\xi_s} \chi_{s\lambda}(y) f(y) dy \\ + \chi_{2\lambda}(x) \frac{1}{\rho_1^2 \omega_1(\lambda)} \int_{\xi_a}^{\xi_1} \phi_{1\lambda}(y) f(y) dy, & x \in (\xi_1, \xi_2) \\ \dots \\ \frac{\chi_{(n+1)\lambda}(x)}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_{\xi_n}^x \phi_{(n+1)\lambda}(y) f(y) dy + \frac{\phi_{(n+1)\lambda}(x)}{\rho_{n+1}^2 \omega_{n+1}(\lambda)} \int_x^b \chi_{(n+1)\lambda}(y) f(y) dy \\ + \chi_{(n+1)\lambda}(x) \sum_{s=1}^n \frac{1}{\rho_s^2 \omega_s(\lambda)} \int_{\xi_{s-1}}^{\xi_s} \phi_{s\lambda}(y) f(y) dy, & x \in (\xi_n, b) \end{cases} \quad (23)$$

Let us introduce the Green's function as

$$G(x, y; \lambda) = \begin{cases} \frac{\phi_{i\lambda}(y) \chi_{i\lambda}(x)}{\omega_i(\lambda)}, & a < y \leq x < b \quad x, y \neq \xi_i = 1, 2, \dots, n \\ \frac{\phi_{i\lambda}(x) \chi_{i\lambda}(y)}{\omega_i(\lambda)}, & a < x \leq y < b \quad x, y \neq \xi_i = 1, 2, \dots, n \end{cases} \quad (24)$$

Then from (23) and (24) it follows that the considered problem (18), (19) has an unique solution given by

$$u(x, \lambda) = \int_a^b G(x, y; \lambda) f(y) dy \quad (25)$$

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